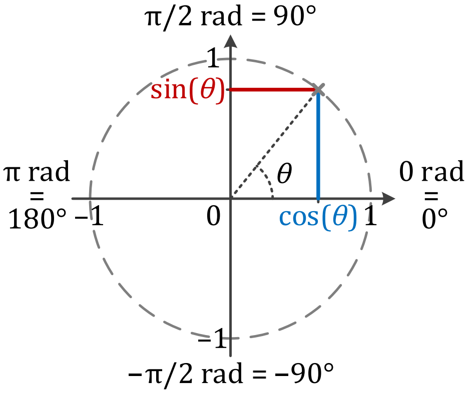
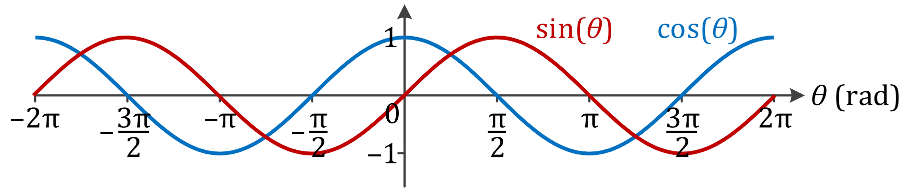
Sine & Cosine

Definitions





θ (rad)	0 (0°)	π/6 (30°)	π/4 (45°)	π/3 (60°)	π/2 (90°)
$\cos(\theta)$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
$\sin(\theta)$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

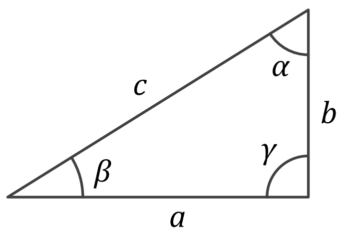
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots$$

$$\cos(x) = \mathrm{Re}\left(e^{ix}\right) = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin(x) = \mathrm{Im}\left(e^{ix}\right) = \frac{e^{ix} - e^{-ix}}{2i} = \frac{e^{-ix} - e^{ix}}{2} i$$

$$e^{ix} = \sum_{n=0}^{\infty} \frac{ix^n}{n!} = \cos(x) + i\sin(x)$$

Link with triangles



For any triangle:	$c^2 = a^2 + b^2 - 2ab\cos(\gamma)$
For a rectangle triangle ($\gamma = \pi/2$):	$c^2 = a^2 + b^2$
cos: adjacent / hypotenuse	$\cos(\alpha) = \frac{b}{c}$ $\cos(\beta) = \frac{a}{c}$
sin: opposite / hypotenuse	$\sin(\alpha) = \frac{a}{c}$ $\sin(\beta) = \frac{b}{c}$

Properties

$$\cos^2(x) + \sin^2(x) = 1 \qquad \cos(-x) = \cos(x) \qquad \sin(-x) = -\sin(x)$$

$$\cos(x \pm 2\pi k) = \cos(x),\ k \in \mathbb{Z} \qquad \cos(x \pm \pi) = -\cos(x) \qquad \cos(x \pm \pi/2) = \mp\sin(x)$$

$$\sin(x \pm 2\pi k) = \sin(x),\ k \in \mathbb{Z} \qquad \sin(x \pm \pi) = -\sin(x) \qquad \sin(x \pm \pi/2) = \pm\cos(x)$$

Sums & products

$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b) \qquad \cos^2(a) = \frac{1 + \cos(2a)}{2}$$

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b) \qquad \sin^2(a) = \frac{1 - \cos(2a)}{2}$$

$$\cos(a) + \cos(b) = 2\cos\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right) \qquad \cos(a)\cos(b) = \frac{\cos(a-b) + \cos(a+b)}{2}$$

$$\cos(a) - \cos(b) = -2\sin\left(\frac{a-b}{2}\right)\sin\left(\frac{a+b}{2}\right) \qquad \sin(a)\sin(b) = \frac{\cos(a-b) - \cos(a+b)}{2}$$

$$\sin(a) + \sin(b) = 2\cos\left(\frac{a-b}{2}\right)\sin\left(\frac{a+b}{2}\right) \qquad \sin(a)\cos(b) = \frac{\sin(a+b) + \sin(a-b)}{2}$$

$$\sin(a) - \sin(b) = 2\sin\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right) \qquad \cos(a)\sin(b) = \frac{\sin(a+b) - \sin(a-b)}{2}$$

Combinations

$$a\cos(\omega t) + b\sin(\omega t) = c\cos(\omega t + \varphi) \Leftrightarrow c = \sqrt{a^2 + b^2} \quad \varphi = \mathrm{atan2}(-b, a) \quad a = c\cos(\varphi) \quad b = -c\sin(\varphi)$$

$$a\cos(\omega t) + b\sin(\omega t) = c\sin(\omega t + \varphi) \Leftrightarrow c = \sqrt{a^2 + b^2} \quad \varphi = \mathrm{atan2}(a, b) \quad a = c\sin(\varphi) \quad b = c\cos(\varphi)$$

$$a\cos(\omega t + \varphi_1) + b\cos(\omega t + \varphi_2) = c\cos(\omega t + \varphi)$$

$$a\sin(\omega t + \varphi_1) + b\sin(\omega t + \varphi_2) = c\sin(\omega t + \varphi)$$

$$\Leftrightarrow \quad c = \sqrt{a^2 + b^2 + 2ab\cos(\varphi_1 - \varphi_2)}, \quad \varphi = \mathrm{atan2}\left(a\sin(\varphi_1) + b\sin(\varphi_2),\ a\cos(\varphi_1) + b\cos(\varphi_2)\right)$$

Derivatives & integrals

$$\frac{d}{dt}\cos(\omega t + \varphi) = -\omega\sin(\omega t + \varphi) \qquad \frac{d}{dt}\sin(\omega t + \varphi) = \omega\cos(\omega t + \varphi) \qquad \frac{d}{dt}e^{i(\omega t + \varphi)} = i\omega\, e^{i(\omega t + \varphi)}$$

$$\int \cos(\omega t + \varphi)\,dt = \frac{\sin(\omega t + \varphi)}{\omega} + C \qquad \int \sin(\omega t + \varphi)\,dt = -\frac{\cos(\omega t + \varphi)}{\omega} + C \qquad \int e^{i(\omega t + \varphi)}dt = \frac{e^{i(\omega t + \varphi)}}{i\omega} + C$$